Scaling limits for subcritical planar Laplacian growth models

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ALE(α, η) – a family of Laplacian growth models

We will formulate a class of planar growth processes, whose state K_t at time $t \geqslant 0$ is a compact set in the plane, starting from the closed unit disk K_0, growing by discrete jumps, and such that $D_t = (\mathbb{C} \cup \{\infty\}) \setminus K_t$ remains simply connected.

It is convenient to encode K_t via the unique conformal isomorphism $\Phi_t : D_0 \to D_t$ such that $\Phi_t(\infty) = \infty$ and $\Phi_t'(\infty) > 0$.

The data for ALE(α, η), besides the two parameters $\alpha, \eta \in \mathbb{R}$, are a choice of capacity parameter $c \in (0, \infty)$ which determines the scale of the individual particles added in each jump, a choice of regularization parameter $\sigma \in (0, \infty)$ which determines the scale of feedback in the model, and a choice of a family of single-particle maps $(F_c : c \in (0, \infty))$, where F_c corresponds to a particle of capacity c.

Examples of particle families are slits or disks.

ALE(α,η)

ALE(α,η) is a Markov chain $\Phi = (\Phi_t)_{t\geqslant 0}$ of conformal maps

$$\Phi_t : D_0 \to D_t \subseteq D_0, \quad D_0 = \{|z| > 1\}$$

Φ jumps from ϕ to $\phi \circ F_{c(\theta,\phi),\theta}$ at rate $\lambda(\theta,\phi)d\theta$ for $\theta \in [0,2\pi]$

$$F_{c,\theta}(z) = e^{i\theta}F_c(e^{-i\theta}z), \quad F_c(z) = e^c\left(z + \sum_{k=0}^{\infty} a_k(c)z^{-k}\right)$$

$$c(\theta,\phi) = c|\phi'(e^{\sigma+i\theta})|^{-\alpha}, \quad \lambda(\theta,\phi) = c^{-1}|\phi'(e^{\sigma+i\theta})|^{-\eta}$$

Particles eg small sticks or disks, diameter $\sim \sqrt{c}$

$$|\phi'(e^{i\theta})| = \frac{d\ell}{d\theta} = \frac{d(\text{arc length})}{d(\text{harmonic measure})}$$

We take as initial state $\Phi_0(z) = z$.

ALE(α, η)

Φ jumps from ϕ to $\phi \circ F_{c(\theta,\phi),\theta}$ at rate $\lambda(\theta, \phi)$ for $\theta \in [0, 2\pi]$

$$c(\theta, \phi) = c|\phi'(e^{\sigma+i\theta})|^{-\alpha}, \quad \lambda(\theta, \phi) = c^{-1}|\phi'(e^{\sigma+i\theta})|^{-\eta}$$

The cluster $K_t = \mathbb{C} \setminus \Phi_t(D_0)$ is uniquely encoded by the map Φ_t.

The effect on the current cluster K of a jump from ϕ to $\phi \circ F_{c,\theta}$ is to add to K the small set $\phi(e^{i\theta} P_c)$, where P_c is the particle added to K_0 by F_c.

We study the behaviour of K_t in the limit $c \to 0$, $\sigma \to 0$.

Fluid limit? – the LK(ζ) equation

$$\dot{\phi}_t(z) = z\phi_t'(z)\int_0^{2\pi}\frac{1}{2\pi}\frac{z+e^{i\theta}}{z-e^{i\theta}}|\phi_t'(e^{\sigma+i\theta})|^{-\zeta}d\theta$$

This is the radial Loewner–Kufarev equation in D_0 with feedback through the driving measure

$$\mu_t(d\theta) = |\phi_t'(e^{\sigma+i\theta})|^{-\zeta}d\theta.$$

Formally it should describe the fluid limit for a wide range of planar Laplacian growth models: eg DLA $\zeta = 2$, Eden model $\zeta = 1$, dielectric breakdown model $\zeta \geqslant 1$.

- $\zeta = 0$ solution $\phi_t(z) = \phi_0(e^t z)$
- $\zeta = 2, \sigma = 0$ (Hele–Shaw flow) has nice algebraic structure (via $|z|^2 = z\bar{z}$) and a well-developed theory
- otherwise poorly understood

LK(ζ)

$$\dot{\phi}_t(z) = z\phi_t'(z) \int_0^{2\pi} \frac{1}{2\pi} \frac{z + e^{i\theta}}{z - e^{i\theta}} |\phi_t'(e^{\sigma + i\theta})|^{-\zeta} d\theta$$

- local existence and uniqueness for analytic solutions holds for analytic initial data
- disk solutions

$$\phi_t(z) = e^{\tau_t} z, \quad \dot{\tau}_t = e^{-\zeta\tau_t}, \quad \tau_t = \zeta^{-1} \log(1 + \zeta t)$$

- expect good behaviour for $\zeta \leqslant 1$ but fractal / turbulent behaviour for $\zeta > 1$

ALE(α, η) is a stochastic discretized version of LK($\alpha + \eta$).

Does LK(ζ) describe the scaling limit as $c, \sigma \to 0$ of ALE(α, η) in the subcritical case $\zeta = \alpha + \eta \leqslant 1$?

Fractal behaviour is observed in simulations of ALE when $\zeta > 1$.

Result

Theorem (Comm. Math. Phys. 2024)

Let $(K_t)_{t \geqslant 0}$ be an ALE(α, η) starting from the unit disk.

Assume that $\zeta = \alpha + \eta \leqslant 1$ and that $c, \sigma \to 0$ with

- $\sigma \gg c^{1/2}$ *in the case* $\zeta < 1$
- $\sigma \gg c^{1/3}$ *in the case* $\zeta = 1$.

Then K_t converges weakly to a disk of radius e^{τ_t} for all $t \geqslant 0$.

Suppose further that

- $\sigma \gg c^{1/4}$ *in the case* $\zeta < 1$
- $\sigma \gg c^{1/5}$ *in the case* $\zeta = 1$.

Then $(\Phi_t(z) - e^{\tau_t} z)/\sqrt{c}$ converges weakly to an explicit Gaussian limit.

Interpolation formula for fluid limits

$(X_t)_{t \geqslant 0}$ a Markov chain, state-space E, jump kernel $\lambda(x, dy)$

$\dot{x}_t = b(x_t)$, where b is a vector field on E

We use

- the linear map $P_{ts} v_s = v_t$ where $\dot{v}_t = \nabla b(x_t) v_t$ for $t \geqslant s$
- the compensated jump measure $\tilde{\mu}^X$ of X
- the drift β of X.

Assume that $X_0 = x_0$. Then $X_t - x_t = M_t + A_t$ where

$$M_t = \int_{E \times (0,t]} P_{ts}(y - X_{s-}) \tilde{\mu}^X(dy, ds)$$

$$A_t = \int_0^t P_{ts}(\beta(X_s) - b(x_s) - \nabla b(x_s)(X_s - x_s)) ds.$$

[Compute the martingale decomposition of $(P_{ts}(X_s - x_s))_{0 \leqslant s \leqslant t}$.]

State-space and norms for ALE

$$\Phi_t(z) = e^{\mathcal{T}_t}\hat{\Phi}_t(z), \quad \hat{\Phi}_t(z) = z + \sum_{k=0}^{\infty} a_k(t) z^{-k}$$

We take as state variables

$$(\mathcal{T}_t, \Psi_t) \in E = \mathbb{R} \times \mathcal{H}(D_0)$$

where $\Psi_t(z) = \hat{\Phi}_t(z) - z$ and $\mathcal{H}(D_0)$ is the set of holomorphic functions on D_0 bounded at ∞.

$$\|\psi\|_{p,r} = \left(\frac{1}{2\pi} \int_0^{2\pi} |\psi(re^{i\theta})|^p d\theta \right)^{1/p}$$

For $\rho \in (1, r)$ and $D\psi(z) = z\psi'(z)$,

$$\|\psi\|_{\infty,r} \leqslant \left(\frac{\rho}{r-\rho} \right)^{1/p} \|\psi\|_{p,\rho}$$

$$\|D\psi\|_{p,r} \leqslant C \left(\frac{\rho}{r-\rho} \right) \|\psi\|_{p,\rho}.$$

L^p-estimates for multiplier operators

$$M\psi(z) = \sum_{k=0}^{\infty} m(k)\psi_k z^{-k}, \quad \psi(z) = \sum_{k=0}^{\infty} \psi_k z^{-k}.$$

An easy calculation shows that

$$\|M\psi\|_{2,r} \leqslant \sup_k |m(k)| \|\psi\|_{2,r}.$$

Marcinkiewicz's multiplier theorem gives a similar estimate for $p \geqslant 2$. Suppose

$$|m(0)| \leqslant A(M), \quad \sum_{k=0}^{\infty} |m(k+1) - m(k)| \leqslant A(M).$$

For all $p \geqslant 2$, there is a constant $C = C(p) < \infty$ such that

$$\|M\psi\|_{p,r} \leqslant CA(M)\|\psi\|_{p,r}.$$

Linearization of LK(ζ) around a disk solution

$$b(\phi)(z) = D\phi(z)\int_0^{2\pi}\frac{1}{2\pi}\frac{z+e^{i\theta}}{z-e^{i\theta}}|\phi'(e^{\sigma+i\theta})|^{-\zeta}d\theta$$

Consider the first variation equation $\dot{\psi}_t = \nabla b(\phi_t)\psi_t$ along the disk solution $\phi_t(z) = e^{\tau_t}z$ for variations $\psi \in \mathcal{H}(D_0)$. We compute

$$\nabla b(\phi_t)\psi(z) = -Q\psi(z)\dot{\tau}_t$$

where, for $q(k) = k(1-\zeta e^{-\sigma(k+1)})$,

$$Q\psi(z) = \sum_{k=0}^{\infty} q(k)\psi_k z^{-k}, \quad \psi(z) = \sum_{k=0}^{\infty}\psi_k z^{-k}.$$

So, for $s \leqslant t$,

$$\psi_t = P_{ts}\psi_s = P(\tau_t - \tau_s)\psi_s, \quad P(\tau) = e^{-\tau Q}.$$

Marcinkiewicz gives the useful bounds

$$\|DP(\tau)\psi\|_{p,r} \leqslant \begin{cases} C(p)/((1-\zeta)\tau), & \zeta < 1 \\ C(p)/(\tau \wedge (\sigma\tau)^{1/2}), & \zeta = 1. \end{cases}$$

L^p-estimates for martingales

Burkholder's inequality states that, for all $p \geqslant 2$, there is a constant $C(p) < \infty$ such that, for all martingales M and all $t \geqslant 0$,

$$\|M_t^*\|_p \leqslant C(p) \left(\|\langle M \rangle_t\|_{p/2}^{1/2} + \|(\Delta M)^*\|_p \right).$$

Here

$$M_t^* = \sup_{s \leqslant t} |M_s|, \quad (\Delta M)_t^* = \sup_{s \leqslant t} |\Delta M_s|.$$

When M has the form

$$M_t = \int_{(0,t] \times E} H(s,y) \tilde{\mu}(ds, dy)$$

for H previsible and $\tilde{\mu}$ a compensated Poisson random measure of intensity $ds \otimes \lambda(dy)$, the terms on the right are given by

$$\langle M \rangle_t = \int_0^t \int_E |H(s,y)|^2 \lambda(dy) ds$$

$$(\Delta M)_t^* \leqslant \sup_{s \leqslant t, y \in E} |H(s,y)|.$$

Fluid limit interpolation in function spaces

$(X_t)_{t \geqslant 0}$ a Markov chain, state-space $C(E)$ say

X jumps from $x = (x(z) : z \in E)$ by $\Delta(x, \theta) \in C(E)$ at rate $\lambda(x, \theta)d\theta$ for $\theta \in [0, 2\pi]$

$\dot{x}_t = b(x_t)$, where b is a vector field on $C(E)$

We can write X in terms of a random measure μ on $[0, 2\pi] \times (0, \infty)$ with previsible compensator $\lambda(X_{t-}, \theta)d\theta dt$

$$X_t(z) = x_0(z) + \int_{[0,2\pi]\times(0,t]} \Delta(X_{s-}, \theta)(z)\mu(d\theta, ds).$$

Then the interpolation formula takes the form $X_t - x_t = M_t + A_t$ where

$$M_u(z) = \int_{[0,2\pi]\times(0,u]} P_{ts}\Delta(X_{s-}, \theta)(z)\tilde{\mu}(d\theta, ds)$$

$$A_t(z) = \int_0^t P_{ts}(\beta(X_s) - b(x_s) - \nabla b(x_s)(X_s - x_s))(z)ds.$$

Estimation of $M_t(z)$ in $L^p(E, dz)$

Apply Burkholder's inequality to $(M_u(z))_{u \leqslant t}$ for each z and then integrate to obtain

$$\||\, M_t \||_p := \left(\mathbb{E} \int_E |M_t(z)|^p dz \right)^{1/p} = \| \|M_t\|_{L^p(\Omega)} \|_{L^p(E)}$$

$$\leqslant C(p) \left(\||\langle M(.)\rangle_t \,\||_{p/2}^{1/2} + \||\, (\Delta M(.))_t^* \||_p \right).$$

Now

$$\langle M(z)\rangle_t = \int_0^t \int_0^{2\pi} |P_{ts}\Delta(X_s, \theta)(z)|^2 \lambda(X_s, \theta) d\theta ds$$

so

$$\|\langle M(.)\rangle_t\|_{L^{p/2}(E)} \leqslant \int_0^t \int_0^{2\pi} \|P_{ts}\Delta(X_s, \theta)\|_{L^p(E)}^2 \lambda^*(\theta) d\theta ds$$

and so

$$\||\langle M(.)\rangle_t\||_{p/2} \leqslant \int_0^t \|P_{ts}\|_{p\to p}^2 ds \int_0^{2\pi} \|\Delta^*(\theta)\|_p^2 \lambda^*(\theta) d\theta.$$

We would like to understand ...

Dynamics for the LK(ζ) equation, especially the case $\zeta = 1$. What is the domain of attraction for disks?

Limits for the case $\zeta = 1$ and $\sigma = c^{1/3}$. For a closely related model, formally, we see convergence of fluctuations to the KPZ equation on long time-scales.

For $\zeta > 1$ the LK(ζ) equation fails to capture the dynamics. So What does the driving measure for ALE look like when $\zeta > 1$ and c is small?